On Sub-Propositional Fragments of Modal Logics

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Menu

1. Clausal form of Modal Logic;
2. Horn/Krom fragments, $\Diamond,\Box$-fragments, and combinations;
3. at **GANDALF 2016** (Catania, Italy, September 2016):
   3.1 Weak and strong relative expressive power;
   3.2 Examples of results and current picture.
4. at **TIME 2016** (Lyngby, Denmark, October 2016)[1]
   4.1 Linear model property for sub-Horn fragments of Modal Logic;
   4.2 Complexity results for sub-Horn fragments of Modal Logic.

[1] or at the **2nd Project Meeting of the INdAM-GNCS Project 2016 "Logic, Automata and Games for Self-Adaptive Systems"**, co-located with GANDALF, Friday afternoon

Motivation

- Modal Logic is propaedeutical to a huge variety of formal systems that we use to model, for example, time and space, and to reason about them.
- Sub-propositional Modal Logic has received a very little attention (Chen,Lin 1994, Del Cerro, Pentotten 1987, Nguyen 2004); but, more recently, some temporal and description logics have been studied under this point of view.
- What happens to the sub-propositional fragments of basic Modal Logic? What is their expressivity and complexity status? And of some natural axiomatic extensions?

Some Syntax

We consider the classical modal language

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi$$

We define positive literals as

$$\lambda ::= \top \mid p \mid \Diamond\lambda \mid \Box\lambda,$$

and we say that a modal formula is in clausal form if it can be produced by the following grammar:

$$\varphi ::= \lambda \mid \neg\lambda \mid \nabla(\neg\lambda_1 \vee \neg\lambda_2 \vee \ldots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \lambda_{n+2} \vee \ldots \vee \lambda_{n+m}) \mid \varphi \wedge \varphi,$$

where $\nabla = \underbrace{\Box\Box\ldots}_{s}$ and $s \geq 0$.

Some Syntax

In this way sub-propositional fragments of the language can be defined in a very natural way:

- Horn: at most one positive literal; $\nabla(\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda)$
- Krom: at most two literals; $\nabla(\lambda_1 \vee \lambda_2), \nabla(\lambda_1 \rightarrow \lambda_2)$
- core: Horn+Krom. $\nabla(\lambda_1 \rightarrow \lambda_2)$

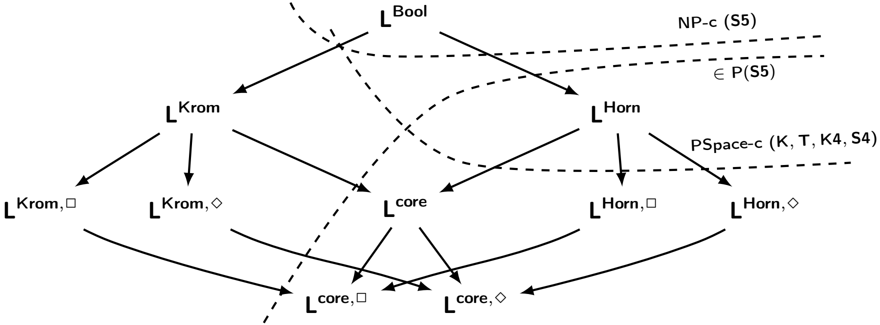
In addition, we can limit the use of $\Diamond$ or $\Box$ in positive literals:

- $\Box$-fragment: $\lambda ::= \top \mid p \mid \Box\lambda$;
- $\Diamond$-fragment: $\lambda ::= \top \mid p \mid \Diamond\lambda$.

We are interested in **K**, **T** (reflexive), **K4** (transitive), **S4** (reflexive and transitive). We mention **S5** (transitive, reflexive and symmetric) for comparison.

Some Syntax

By combining these two forms of restrictions, we obtain the following grammatical containment Hasse diagram:



- If this is TIME 2016, click here!
- Otherwise, keep on!

Relative Expressive Power

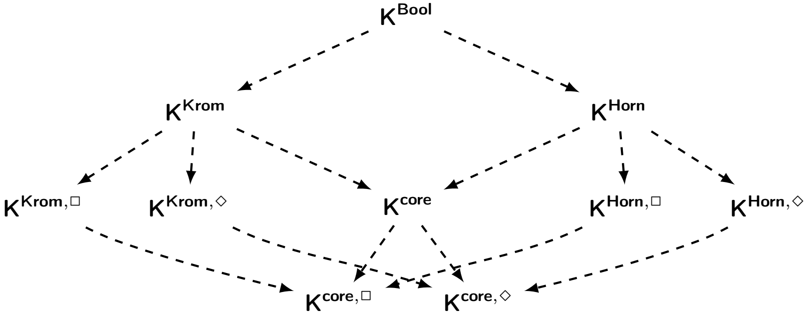
- How do we measure the relative expressive power among different fragments?
- Weak expressivity: any formula $\varphi \in \mathbf{L}$ can be translated to an equivalent formula $\varphi' \in \mathbf{L}'$ written with the same propositional alphabet. We denote this situation with $\mathbf{L} \preceq^w \mathbf{L}'$.
  - Example: $p \to (q \wedge r)$ can be translated to $(p \to q) \wedge (p \to r)$
- Strong expressivity: any formula $\varphi \in \mathbf{L}$ can be translated to an equivalent formula $\varphi' \in \mathbf{L}'$ possibly by adding new propositional letters. We denote this situation with $\mathbf{L} \preceq \mathbf{L}'$.
  - Example: $\Box\Box p$ can be translated to $\Box q \wedge \Box(q \to \Box p)$

Relative Expressive Power

- The weak notion deals with relative expressive power within the same propositional alphabet.
- The strong notion allows one to add as (finitely) many new propositional letters as they are needed in order to translate a formula of a language to a formula of another language. In this sense it is strong: when we give a negative result, then no matters how many letters you add, there is a formula you cannot translate.
- Strong results are difficult to obtain: one has to quantify over all possible extensions!
- If this is the GNCS meeting, click here!
- Let us now focus on **K**.

Weak Results

The following results hold:



Weak Results

As an example, we prove $\mathsf{K}^{\mathbf{Horn}} \prec^{w} \mathsf{K}^{\mathbf{Bool}}$.

Proof. The following steps do the trick:

- $\mathsf{K}^{\mathbf{Horn}}$ is a syntactical fragment of $\mathsf{K}^{\mathbf{Bool}}$, so $\mathsf{K}^{\mathbf{Horn}} \preceq^{w} \mathsf{K}^{\mathbf{Bool}}$;
- Consider the $\mathsf{K}^{\mathbf{Bool}}$-formula

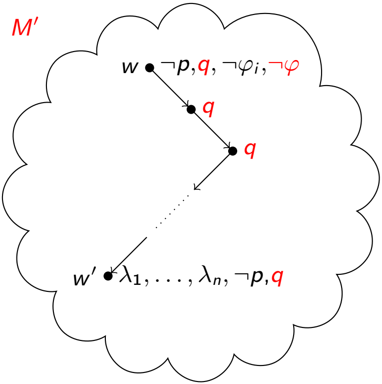
$$\psi \equiv p \vee q,$$

  and prove by contradiction that no conjunction of $\mathsf{K}^{\mathbf{Horn}}$-clauses may translate it assuming that the propositional alphabet is the same!
- By contradiction, let $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_l$ be such that each φ_i is a Horn clause, and:

$$M, w \Vdash \psi \Leftrightarrow M, w, \Vdash \varphi \text{ for every model } M \text{ and world } w.$$

Weak Results

Proof (cont'ed).



- Take M s.t. $M, w \not\Vdash p \vee q$
- hence, $M, w \not\Vdash \varphi_i$, that is, $M, w \not\Vdash \Box^s(\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda)$
- we can find w' such that $M, w' \not\Vdash \lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda$
- suppose $\lambda = p$
- make q true everywhere on the model
- now we have that $M', w \Vdash p \vee q$ but $M', w' \not\Vdash \lambda_1 \wedge \ldots \lambda_n \rightarrow \lambda$
- $M', w \not\Vdash \varphi$: contradiction!

Weak Results

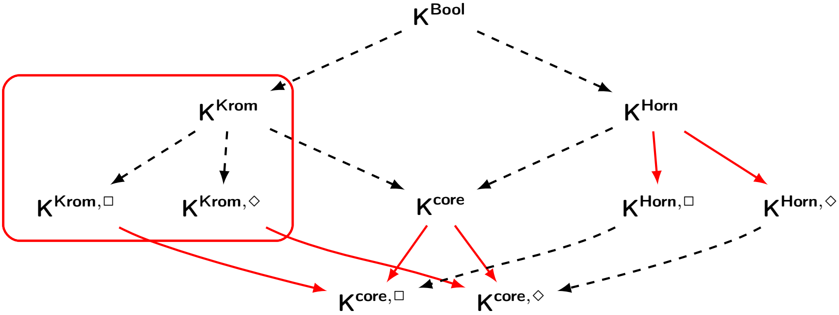
Proof (cont'ed).

- By a similar reasoning, we can easily prove that $M', w \not\Vdash \varphi_i$ for all possible λ.
- In all cases we have a contradiction with the assumption that φ is a translation of $p \vee q$. $\square$

Our 'weak' hypothesis (building φ using p and q only) played an essential role. The other weak results are similar (but not identical).

Strong Results

The following results hold:



Positive Strong Result

As an example, prove that $\mathsf{K}^{\mathbf{Krom},\Box}$ is strongly equivalent to $\mathsf{K}^{\mathbf{Krom}}$.

Proof. Let us sketch it as follows:

- $\mathsf{K}^{\mathbf{Krom},\Box}$ is a syntactical fragment of $\mathsf{K}^{\mathbf{Krom}}$, so $\mathsf{K}^{\mathbf{Krom},\Box} \preceq \mathsf{K}^{\mathbf{Krom}}$.
- Consider a $\mathsf{K}^{\mathbf{Krom}}$-clause $\nabla(\Diamond\lambda_1 \vee \lambda_2)$
- it is equivalent to the $\mathsf{K}^{\mathbf{Krom},\Box}$-formula
  $$\nabla(\neg\Box p \vee \lambda_2) \wedge \nabla\Box(p \vee \lambda_1)$$
  where p is a fresh propositional variable.
- the case for $\neg\Diamond\lambda_1$ is similar. □

Negative Strong Results

As an example, let us prove that $\mathsf{K}^{\mathbf{Horn},\Box} \prec \mathsf{K}^{\mathbf{Horn}}$.

Proof. Let us sketch it as follows:

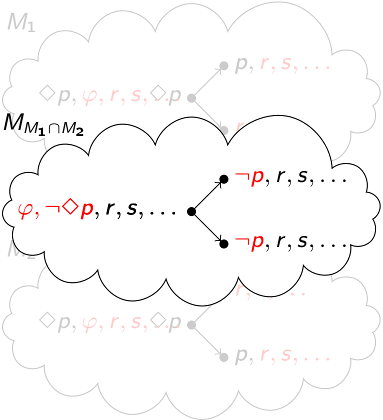
- $\mathsf{K}^{\mathbf{Horn},\Box}$ is a syntactical fragment of $\mathsf{K}^{\mathbf{Horn}}$, so $\mathsf{K}^{\mathbf{Horn},\Box} \preceq \mathsf{K}^{\mathbf{Horn}}$.
- Consider the $\mathsf{K}^{\mathbf{Horn}}$-formula

$$\psi \equiv \Diamond p,$$

  and prove by contradiction that no conjunction of $\mathsf{K}^{\mathbf{Horn},\Box}$-clauses may translate no matters how you extend the propositional alphabet!
- The proof exploits the fact that $\mathsf{K}^{\mathbf{Horn},\Box}$ is closed under intersection of models (based on the same relational frame).

Negative Strong Results

Proof (cont'ed).



- Suppose $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_l$ is a $\mathbf{K}^{\mathbf{Horn},\Box}$-formula equivalent to $\Diamond p$
- consider the models M_1 and M_2
- we can turn them into models for φ by adding the valuation of the new propositional letters
- build the intersection model $M_{M_1 \cap M_2}$
- $M_{M_1 \cap M_2}, w \not\Vdash \Diamond p$
- $M_{M_1 \cap M_2}, w \Vdash \varphi$ (by the closure property);
- contradiction! □

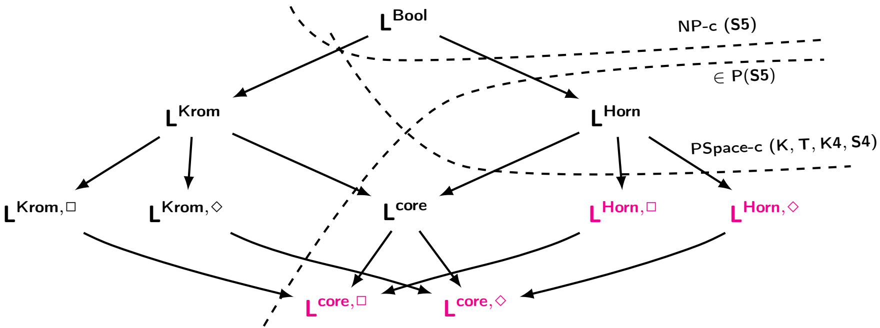
Conclusions (of the Expressive Power Part)

- We know the relative expressive power of the sub-propositional **K**-fragments:
  - the picture for the strong expressivity is still incomplete
- Our techniques can be applied (with some adjustments) to axiomatic extensions of **K**, to point-based temporal logics (such as LTL) or interval-based temporal logics (such as HS), in order to obtain a clear picture of their relative expressive power.
- What about complexity?

To be continued...

Complexity Results - General View

From the results presented at GANDALF 2016 we understood that some fragments of **K** and its axiomatic extensions may be interesting from the complexity point of view. Let $\mathbf{L} \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}\}$:



Complexity Results - General View

- This depends on the proven fact that there is a 'strong' expressive power jump from the the original language (and from the Horn fragment).
- Let us consider now the easiest case of all: K under the Horn Box restriction (i.e.: only clauses with no diamonds!).
- The intuition behind some improvement in the complexity of the satisfiability problem is clear; but how to prove it?

Pre-Linear Models

- We say that a relational structure (W, R) is pre-linear if it is a tree of arity 1, namely, if:
  (i) there exists a unique node w_0 (called root) such that, for every $w \in W$, $w_0 R^* w$ (where R^* is the reflexive and transitive closure of R);
  (ii) every $w \in W$ distinct from w_0 has a unique R-predecessor;
  (iii) every $w \in W$ has at most one R-successor; and
  (iv) the transitive closure of R (denoted by $\overrightarrow{R}$) is acyclic.
- In a pre-linear structure we can enumerate the worlds in W as $w_0, w_1, \ldots$, where w_0 is the root and for each $k \geq 0$ we have that w_{k+1} is the unique R-successor of w_k. *Pre-linear models* are models built on pre-linear structures.

Pre-Linear Models

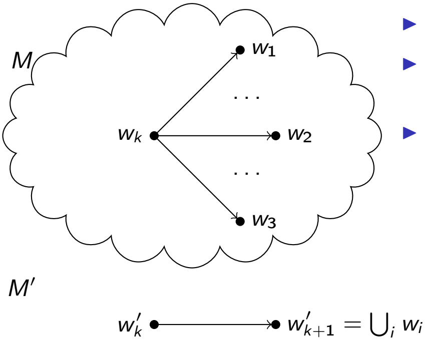
Let φ be a $\mathsf{K}^{\mathbf{Horn},\Box}$-formula. Then, we can prove that φ is satisfiable if and only if it is satisfiable in a pre-linear model.

Proof.

- Let φ be a satisfiable $\mathsf{K}^{\mathbf{Horn},\Box}$-formula, and let $M = (\mathcal{F}, V)$, where $\mathcal{F} = (W, R)$, be a model that satisfies it at w_0. We can assume that $\mathcal{F}$ is connected and acyclic. If this is not the case we can take the unfolding of $\mathcal{F}$ from the initial world w_0.
- We now systematically build a pre-linear model M', with set of worlds W', as follows: the worlds in W' are sets of worlds in W. We set $W_0 = \{w'_0\}$, $w'_0 = \{w_0\}$, $R_0 = \emptyset$, and $V_0(w'_0) = V(w_0)$.

Pre-Linear Models

Proof (cont'ed).



- $w_k R w_1, w_k R w_2, w_k R w_3, \ldots$
- $w_{k+1} = \{w_1, w_2, w_3, \ldots\}$
  $V(w_{k+1}) = \bigcap_i V(w_i)$
- $M' = \bigcup_k M_k$ is a pre-linear model. Now, one shows that $M', w'_0 \Vdash \varphi$ □

Pre-Linear Models

- Satisfiability can be reduced to pre-linear models in each of the considered cases, namely the transitive, the reflexive, and the transitive and reflexive case.
- Some modifications to the above proof are necessary.
- In case of $\mathsf{K}^{\mathbf{Horn},\Box}$, the above statement can be improved as follows:

  $\varphi \in \mathsf{K}^{\mathbf{Horn},\Box}$ is satisfiable if and only if it is satisfiable in a pre-linear model M such that $|M| \leq md(\varphi) + 1$,

  where $md()$ is the modal depth of the formula.
- Similar results, but slightly more complex, can be stated in the other three cases.

A P-Time Algorithm for Satisfiability

```
function HornBoxSat(φ)
    W ← {w_0}
    H(w_0) ← {φ_1, . . . , φ_l}
    L(w_0) ← {⊤}
    N = md(φ)
    for d ← 0, . . . , N do
        if Saturate(W, H, L) then
            W ← W ∪ {w_{d+1}}
            H(w_{d+1}) ← ∅
            L(w_{d+1}) ← {⊤}
        else
            return Shorten(W, H, L, d)
    return True
```

```
function Shorten(W, H, L, d)
    B ← {□λ | □λ ∈ Cl(φ)}
    for k ← d, d − 1, . . . , 1 do
        W ← W \ {w_k}
        H ← H|_W, L ← L|_W
        H(w_{k−1}) ← H(w_{k−1}) ∪ B
        if Saturate(W, H, L) then
            return True
    return False
```

```
function Saturate(W, H, L)
    while something changes do
        let w_k ∈ W, ψ ∈ H(w_k)
        if ψ = p then
            H(w_k) ← H(w_k) \ {p}
            L(w_k) ← L(w_k) ∪ {p}
            if w_{k−1} ∈ W and □p ∈ Cl(φ) then
                H(w_{k−1}) ← H(w_{k−1}) ∪ {□p}
        else if ψ = □ξ then
            if w_{k+1} ∈ W then
                H(w_k) ← H(w_k) \ {□ξ}
                L(w_k) ← L(w_k) ∪ {□ξ}
                H(w_{k+1}) ← H(w_{k+1}) ∪ {ξ}
            if w_{k−1} ∈ W and □□ξ ∈ Cl(φ) then
                H(w_{k−1}) ← H(w_{k−1}) ∪ {□□ξ}
        else if ψ = λ_1 ∧ . . . ∧ λ_n → λ, λ ≠ ⊥ then
            if {λ_1, . . . , λ_n} ⊆ H(w_k) ∪ L(w_k) then
                H(w_k) ← (H(w_k) ∪ {λ}) \ {ψ}
                L(w_k) ← L(w_k) ∪ {ψ}
        else if ψ = λ_1 ∧ . . . ∧ λ_n → ⊥ then
            if {λ_1, . . . , λ_n} ⊆ H(w_k) ∪ L(w_k) then
                return False
    return True
```

A P-Time Algorithm for Satisfiability

- The procedure iteratively builds a structure (W, H, L) that represents a candidate model for the formula, where
  - W is the set of worlds of the model,
  - H labels each world in W with the set of formulas 'to be further analyzed', and
  - L labels each world w with the set of formulas in $Cl(\varphi)$ that holds on w.
- If everything is correct during the construction, a new world is added (up to the theoretical limit), and the entire model is saturated.
- Saturation is simply the process of systematically fill up the current model with every consequence of the set of clauses.

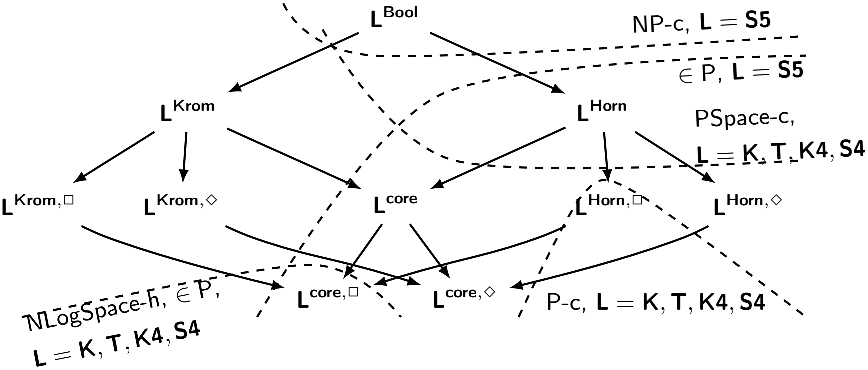
A P-Time Algorithm for Satisfiability

- If, during the saturation, a contradiction is found, then
  - We shorten the candidate model, and
  - We saturate it again, taking into account the new literals that become true from the knowledge that the model is shorter than a certain length.
- Either a model is found in this way, or the shortening procedure 'eats up' all of it, returning that the formula is unsatisfiable.
- Example: the set $\{p, \Box p \rightarrow \bot, \Box p \rightarrow q\}$ is satisfiable with *at least* 2 points.
- Example: the set $\{\Box p, \Box q, \Box((p \wedge q) \rightarrow \bot)\}$ is satisfiable with *at most* 1 point.

A P-Time Algorithm for Satisfiability

- By taking into account the suitable modifications to the above algorithm for the other cases, we can give the following argument to establish the complexity of the fragments Horn Box and core Box:
  - The most external cycle runs $O(|\varphi|)$ times in the worst case.
  - The candidate models grow from 1 to $O(|\varphi|)$ points, and, then, down to 1 again; in the H component of each point there are, at most, $O(|\varphi|)$ formulas on which Saturate has effect.
  - Therefore, the total time spent is $O(|\varphi|) \cdot 2 \cdot \Sigma_{i=1}^{O(|\varphi|)} i = O(|\varphi|^3)$, which is polynomial in $|\varphi|$.
- The Horn Box fragment of all **K**, **T**, **K4**, and **S4** contain the Horn fragment of propositional logic, which allow us to conclude that their satisfiability problem is P-complete.

Conclusions (of the Complexity Part)



Conclusions (of the Complexity Part)

- Among the most relevant open problems in this topic we mention the Horn Diamond and Core Diamond fragment, previously mentioned as potential candidates for a better complexity.
- As in the relative expressive power side of this story, our techniques can be applied to more interesting logics nevertheless the problem is more complex and the results we have shown should be considered more as a 'compass' towards well-behaved modal (temporal, spatial) logics than anything else.
- Besides other interesting axiomatic extensions of **K**, the most promising direction now includes simple fragments of the interval-based temporal logic HS. These tend to be decidable, but at a very high complexity; can we improve it by looking into sub-propositional fragments?