A Canonical Model Construction for Iteration-Free PDL with Intersection

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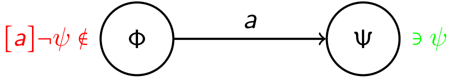
Canonical Models

Tool to show completeness of proof calculus (for e.g., ML)

Idea:

- take set of maximally consistent sets of formulas (mcs) as underlying set of structure
- atomic propositions via membership
- $\Phi \xrightarrow{a} \Psi$ iff $[a]\neg\psi \in \Phi$ for no $\psi \in \Psi$



→ (via induction): φ true at Φ iff $\varphi \in \Phi$.

yields satisfiability of any consistent set of formulas, i.e., completeness.

NB: presence of edge depends only on endpoints.

Iteration-Free PDL with Intersection (PDL_0)

fix propositions $\{P, Q, \dots\} = \mathcal{P}$, atomic programs $\{a, b, \dots\} = \mathcal{R}$

Syntax:

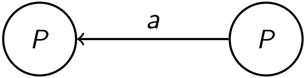
formulas: $\varphi ::= P \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \langle\alpha\rangle\varphi \mid [\alpha]\varphi$

programs: $\alpha ::= a \mid \alpha;\alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \mid \varphi?$

Semantics (sketch) over LTS $\mathcal{T}$:

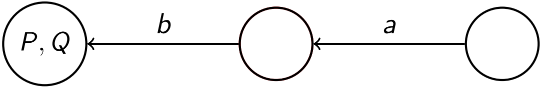
- $\langle\alpha\rangle\varphi$ true at s iff ex. t with $s \xrightarrow{\alpha} t$ and φ true at t
- $s \xrightarrow{a} t$ iff $(s, t) \in a^{\mathcal{T}}$
- $s \xrightarrow{\alpha_1;\alpha_2} t$ iff ex. u with $s \xrightarrow{\alpha_1} u$ and $u \xrightarrow{\alpha_2} t$
- $s \xrightarrow{\alpha_1\cap\alpha_2} t$ iff $s \xrightarrow{\alpha_1} t$ and $s \xrightarrow{\alpha_2} t$
- $s \xrightarrow{\varphi?} t$ iff $s = t$ and φ true at s

$\mathbf{PDL_0}$ in action



$\models P \wedge \langle a \rangle P \wedge [a \cap \top?]\bot$

→ no tree model property



$\not\models \langle a; [b; P?]\neg Q?; b \rangle (P \wedge Q)$

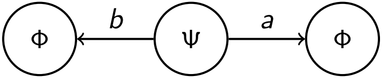
→ convoluted and nested programs hard to conquer inductively

More complications

Consider sat. set Φ (e.g. theory of dead end world)

$\Psi = \bigcup_{\varphi \in \Phi} \{\langle a \rangle \varphi, [a]\varphi, \langle b \rangle \varphi, [b]\varphi\} \cup \{[a \cap b]\bot\}$

Ψ has model:



But no model with only one instance of Φ

→ canonical model needs adaption

Existing constructions not convincing enough

A Proof Calculus for PDL_0

Standard style proof system with derivation rules

$$(\mathtt{MP})\ \frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad (\mathtt{Gen})\ \frac{\varphi}{[\alpha]\varphi} \qquad (\mathtt{USub})\ \frac{\varphi}{\varphi\ulcorner\psi/p\urcorner}$$

$$(\mathtt{PSub})\ \frac{\varphi \qquad \alpha \Rightarrow \alpha'}{\varphi\ulcorner\langle\alpha'\rangle/\langle\alpha\rangle\urcorner}(pos)$$

and axioms and axiom schemes:

$\alpha \cap \beta \Rightarrow \alpha$

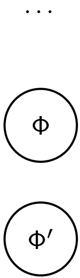
$(p?;\alpha) \cap \beta \Leftrightarrow p?;(\alpha \cap \beta)$

...

Construction of the Canonical Model

Idea: build “free” structure, i.e., maximally tree-like, no unnecessary connections

Construction of the Canonical Model



start with mcs, no edges → atomic and box formulas satisfied (generation 0)

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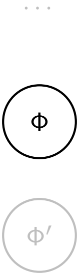

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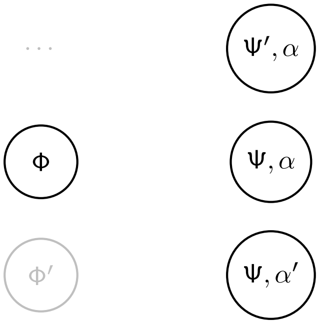
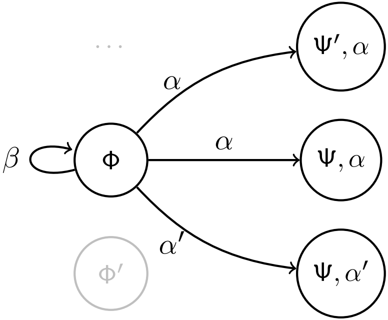


add witnesses for missing diamonds

Construction of the Canonical Model



add witnesses for missing diamonds, connect with abstract edges

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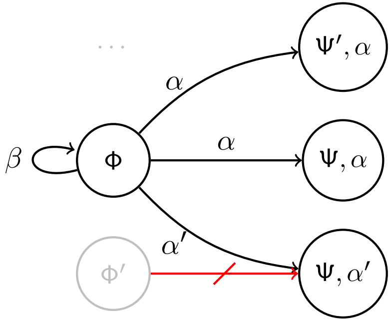

add witnesses for missing diamonds, connect with abstract edges in disjoint fashion

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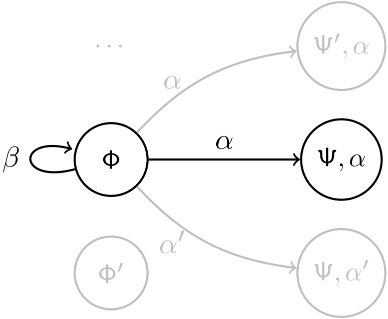

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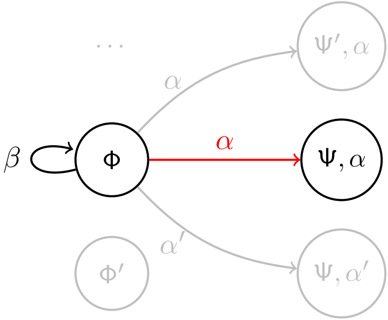

refine iteratively

$\alpha = \alpha_1 \cap \alpha_2$

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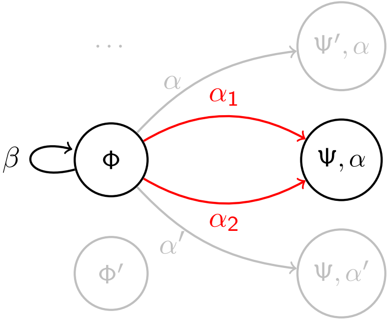

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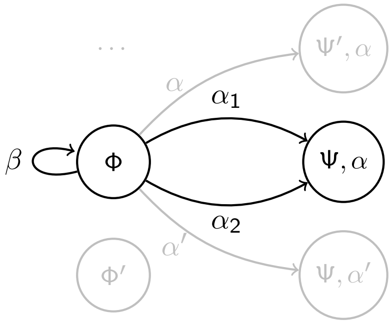

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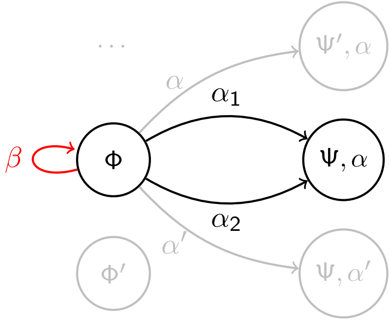

refine iteratively, add intermediate nodes if necessary

$\alpha = \alpha_1 \cap \alpha_2 \qquad \beta = \beta_1 ; \beta_2$

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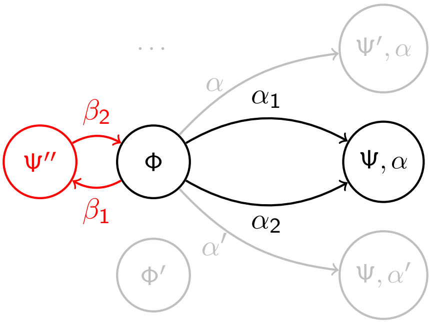

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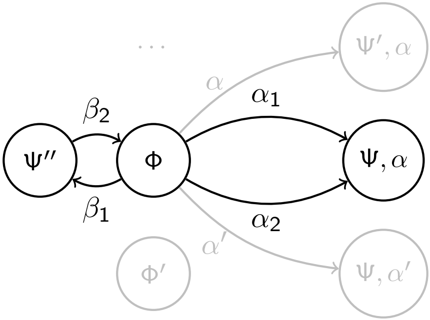
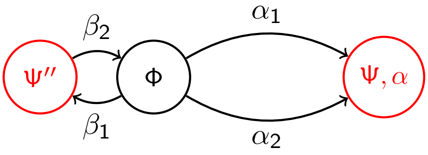


continue inductively until abstract programs converted to concrete programs

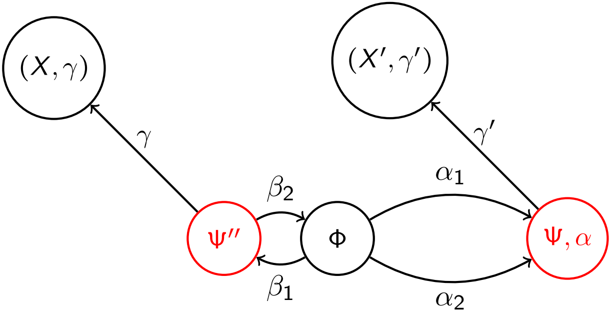
$\alpha = \alpha_1 \cap \alpha_2 \qquad \beta = \beta_1 ; \beta_2$

Construction of the Canonical Model



Problem: New unsatisfied diamonds in generation 1 nodes

Construction of the Canonical Model



Repeat Process: Add witnesses (generation 2), refine

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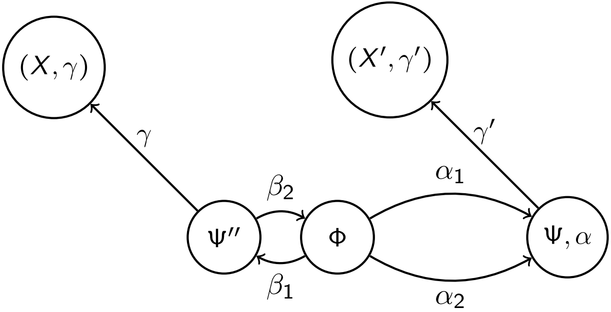
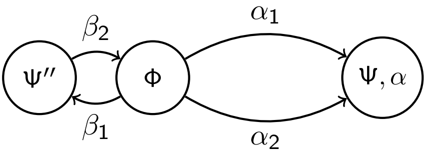


Repeat Process: Add witnesses (generation 2), refine

All diamonds satified in limit (generation ω)

Correctness of the Construction



Need to show: φ true at node labelled Φ iff $\varphi \in \Phi$

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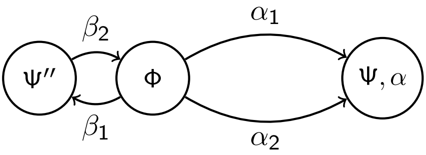

In particular:
If $[\alpha]\neg\psi \in \Psi''$ and $\Psi'' \xrightarrow{\alpha} \Psi$, then $\psi \notin \Psi$

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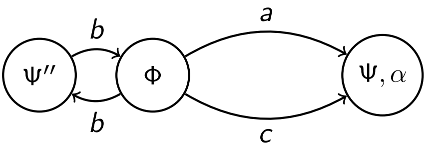

In particular:

If $[(b;a) \cap (b;c)]\neg P \in \Psi''$ and $\Psi'' \xrightarrow{(b;a)\cap(b;c)} \Psi$, then $P \notin \Psi$

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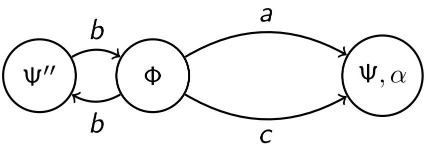

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Problem: Program unplanned: structure constructed for $b;(a \cap c)$

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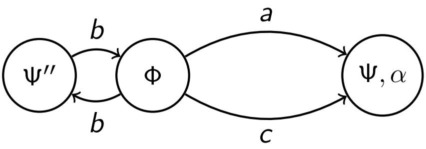

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Problem: Program unplanned: structure constructed for $b;(a \cap c)$
Can rewrite: $[(b;a) \cap (b;c)]\neg P \to [b;(a \cap c)]\neg P$
Correctness of construction provable

End of Talk

Further work:

- Extend to full PDL with intersection, i.e., with Kleene star (weak completeness only)
- Compare present work to existing constructions more thoroughly

Thanks for listening!