Games for topological fixpoint logics

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Gandalf 2016, Catania, 15 September 2016

Overview

- topological semantics of the modal μ-calculus
- fixpoint games
- main result: evaluation game for topological semantics
- bisimulations
- Disclaimer: topological semantics in our setting means
  - Kripke models based on topological spaces
  - semantics of a formula given by an "admissible" subset
  - different from "Spatial Logic of modal mu-calculus" by Goldblatt/Hodkinson (2016)

topological semantics of the modal μ-calculus

Modal μ-calculus

- adding least and greatest fixpoint operators to modal logic
- Goal: express properties about ongoing, (possibly) infinite behaviour

Formulas

$$\begin{aligned}\mathcal{L}_\mu \ni \varphi \quad ::= \quad & \mathrm{p} \mid \neg \mathrm{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \bot \mid \top \mid \Diamond\varphi \mid \Box\varphi \mid \\ & \mu \mathrm{p}.\varphi(\mathrm{p}, \mathrm{q}_1, \ldots, \mathrm{q}_\mathrm{n}) \mid \nu \mathrm{p}.\varphi(\mathrm{p}, \mathrm{q}_1, \ldots, \mathrm{q}_\mathrm{n})\end{aligned}$$

where $\mathrm{p} \in \mathrm{Prop}$ and p occurs only positively in formulas of the form $\mu \mathrm{p}.\varphi(\mathrm{p}, \mathrm{q}_1, \ldots, \mathrm{q}_\mathrm{n})$ and $\nu \mathrm{p}.\varphi(\mathrm{p}, \mathrm{q}_1, \ldots, \mathrm{q}_\mathrm{n})$.

Formulas as operators

For a formula $\delta \in \mathcal{L}_\mu$, a Kripke frame (X, R) and a valuation $\mathrm{V} : \mathrm{Prop} \longrightarrow \mathcal{P}\mathrm{X}$ let

$$\begin{array}{rcl} \delta_p^V : \mathcal{P}X & \longrightarrow & \mathcal{P}X \\ U & \mapsto & [\![\delta]\!]^{(X,R)}_{V[p\mapsto U]} \end{array}$$

where

$$V[p \mapsto U](q) = \begin{cases} U & \text{if } p = q \\ V(q) & \text{otherwise.} \end{cases}$$

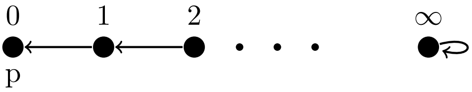
and where $[\![\varphi]\!]^{(X,R)}_V$ denotes semantics of φ on a model (X, R, V).

Observation

If δ contains only positive occurrences of p, then δ_p^V is a monotone operator.

$\Rightarrow$ We define the semantics of $\mu p.\delta$ and $\nu p.\delta$ as least and greatest fixpoints of δ_p^V, respectively.

An example



Standard interpretation:

- R is the transition relation
- $\Diamond^* p = \mu q. p \vee \Diamond q$: p reachable via transitive closure of R
- $\mathbb{N}$ is the least fixed point of the map $U \mapsto V(p) \cup \Diamond U$, where

$$\Diamond U = \{x' \mid \exists x \in U.(x', x) \in R\}$$

With topology:

- basis of clopens: finite subsets of $\mathbb{N}$ and cofinite sets containing ∞
- least clopen fixpoint: $\mathbb{N} \cup \{\infty\}$
- p reachable via topological transitive closure of R

Topological semantics: Motivation

Denotations of formulas restricted to admissible subsets of a model - admissibility defined topologically.

Motivation:

- apply techniques from Stone duality to the μ-calculus
- modal μ-algebras give relatively easy completeness proofs with respect to topological semantics (cf. work by Ambler/Bonsangue/Kwiatkowska)
- completeness wrt standard semantics would follow from finite model property for topological semantics
- notion of reachability in the limit in order to be able to specify systems with continuous behaviour

Our goal

Characterise topological semantics using games, ultimately: automata!

Related Work

- S. Ambler, M. Kwiatkowska, N. Measor. Duality and the Completeness of the Modal mu-Calculus. (1995)
- M.M. Bonsangue, M. Kwiatkowska, Re-Interpreting the Modal mu-Calculus (1995)
- J. van Benthem, N. Bezhanishvili, I. M. Hodkinson. Sahlqvist Correspondence for Modal mu-calculus. (2012)
- N. Bezhanishvili, I. Hodkinson. Sahlqvist theorem for modal fixed point logic. (2012)
- W.Conradie, Y.Fomatati, A.Palmigiano, S.Sourabh, Algorithmic correspondence for intuitionistic modal mu-calculus.(2015)
- S. Enqvist, F. Seifan, Y. Venema, Completeness for the modal μ-calculus: separating the combinatorics from the dynamics. (2016)

Our topological setting

- extremally disconnected Stone spaces: closure of any open set is open (and thus clopen)
- consequently: clopen sets form a complete lattice
- frames: descriptive frames $(\mathbb{X}, \mathrm{R})$ based on an extremally disconnected Stone space $\mathbb{X} = (\mathrm{X}, \tau)$
- model = frame + clopen valuation $\mathrm{V} : \mathrm{Prop} \longrightarrow \mathrm{Clp}(\mathbb{X})$

Formulas interpreted as clopen subsets

- for the "purely modal" formulas this is ensured by taking a clopen valuation and descriptive general frames (=Vietoris coalgebras)

- for the fixpoint operators we note that

  1. $\delta_{\mathrm{p}}^{\mathrm{V}} : \mathrm{Clp}(\mathbb{X}) \to \mathrm{Clp}(\mathbb{X})$ is a monotone operator
  2. $\mathrm{Clp}(\mathbb{X})$ is a complete lattice

- we interpret $\mu\mathrm{p}.\delta$ and $\nu\mathrm{p}.\delta$ as the least and greatest clopen fixpoints, respectively

Fixpoint approximation on a complete lattice

$$\begin{array}{rcll} F^{\mu}_{0} & = & \bot, & \\ F^{\mu}_{\alpha+1} & = & F(F^{\mu}_{\alpha+1}) & \\ F^{\mu}_{\alpha} & = & \bigvee_{\beta<\alpha} F^{\mu}_{\beta} & \text{for } \alpha \text{ a limit ordinal.} \end{array}$$

$$\begin{array}{rcll} F^{\nu}_{0} & = & \top, & \\ F^{\nu}_{\alpha+1} & = & F(F^{\nu}_{\alpha+1}) & \\ F^{\nu}_{\alpha} & = & \bigwedge_{\beta<\alpha} F^{\nu}_{\beta} & \text{for } \alpha \text{ a limit ordinal.} \end{array}$$

Fact

For any monotone operator F there are ordinals α_μ and α_ν such that $F^{\mu}_{\alpha_\mu} = \mu F$ and $F^{\nu}_{\alpha_\nu} = \nu F$.

fixpoint games

A note about games

- all games in my talk are two-player graph games
- players are called $\exists$ and $\forall$ and board positions are partitioned into $\exists$'s and $\forall$'s positions
- plays can be infinite
- the winning condition on infinite games can be encoded using a parity condition
- this means that at any position of the game board either of the player has a historyfree/memoryless winning strategy

Tarski's fixpoint game on $\mathcal{P}X$

Here we have:

$$\bigwedge_{i\in I} U_i = \bigcap_{i\in I} U_i \qquad \text{and} \qquad \bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$$

We define the game board of a two-player graph game:

Position	Player	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(C)\}$
$C \in \mathcal{P}X$	$\forall$	C

Least fixpoint game: $\forall$ wins infinite plays.
Greatest fixpoint game: $\exists$ wins infinite plays.

Fact

$\exists$ has a winning strategy at position $x \in X$ in the least (greatest) fixpoint game iff $x \in \mu F$ ($x \in \nu F$).

Proof idea for least fixpoint game

Suppose $\mathrm{x} \in \mu\mathrm{F}$.

This means there exists an ordinal α such that $\mathrm{x} \in \mathrm{F}^{\mu}_{\alpha}$.

Suppose α is a limit ordinal, then

$$\mathrm{x} \in \bigvee_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta} = \bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta} \subseteq \mathrm{F}(\bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta})$$

Therefore $\exists$ moves from x to $\bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta}$ and $\forall$ has to move to some $\mathrm{x}' \in \mathrm{F}^{\mu}_{\beta}$ with $\beta < \alpha$.

Similarly for successor ordinals.

Well-foundedness of the ordinals implies finiteness of the play.

Proof idea for least fixpoint game

For the converse assume that $x \notin \mu F$.

Then either $\exists$ gets stuck at x, or she moves to some $C \subseteq X$ with $x \in F(C)$.

By assumption we know that $C \not\subseteq \mu F$, ie., $\forall$ can continue the play by moving to some $x' \in C \setminus \mu F$ and the play continues ...

Either the play will be infinite or $\exists$ gets stuck which shows that $\forall$ has a winning strategy.

Fixpoints on an extremally disconnected Stone space

$$\bigwedge_{i \in I} U_i = \mathrm{Int}(\bigcap_{i \in I} U_i) \qquad \text{and} \qquad \bigvee_{i \in I} U_i = \mathrm{Cl}(\bigcup_{i \in I} U_i)$$

Possible consequences for the μ-game (informal!):

1. To show that $x \in \mu F$ it suffices for $\exists$ to provide a "suitable" set $C \subseteq X$ such that $x \in F(U)$ for all clopens $U \in \mathrm{Clp}(\mathbb{X})$ with $C \subseteq U$.
2. If $\exists$ chooses some (open) set $O \subseteq X$ such that $x \in F(\mathrm{Cl}(O))$, $\forall$ is only allowed to challenge her with elements of O.

Both observations lead to different (but equivalent) games.

Topological least fixpoint game I

Position	Player	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U)$ for all $U \in Clp(\mathbb{X})$ with $C \subseteq U\}$
$C \in \mathcal{P}X$	$\forall$	C

Topological least fixpoint game II

Position	Pl.	Moves
$x \in X$	$\exists$	$\{(\forall, U) \in M \times Clp(\mathbb{X}) \mid x \in F(U)\}$
$(\forall, U) \in M \times Clp(\mathbb{X})$	$\forall$	$\{(\exists, U') \in M \times Clp(\mathbb{X}) \mid U \cap U' \neq \emptyset\}$
$(\exists, U') \in M \times Clp(\mathbb{X})$	$\exists$	U'

(Here $M = \{\forall, \exists\}$ is the collection of markers to keep sets of positions of $\exists$ and $\forall$ disjoint.)

Correctness of $\mathcal{G}^{\mathrm{II}}$

Suppose that $x \in \mu F$ for some $x \in X$.

Then there is a least ordinal α such that $x \in F^{\mu}_{\alpha}$.

We will show that $\exists$ has a (winning) strategy that ensures that either $\forall$ gets stuck within the next round or that the play reaches a position $x' \in F^{\mu}_{\alpha'}$ with $\alpha' < \alpha$.

Correctness of $\mathcal{G}^{\mathrm{II}}$

Case α is a limit ordinal

Then $\exists$'s strategy is to move from x to

$$(\forall, \bigvee_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta}) = (\forall, \mathrm{Cl}(\bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta})).$$

Unless $\forall$ gets stuck, he will move to some position $(\exists, \mathrm{U}')$ where $\mathrm{U}' \in \mathrm{Clp}(\mathbb{X})$ with

$$\mathrm{U}' \cap \bigvee_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta} = \mathrm{U}' \cap \mathrm{Cl}(\bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta}) \neq \emptyset.$$

One can easily see that this implies $\mathrm{U}' \cap \bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta} \neq \emptyset$. Therefore $\exists$ can pick a suitable element $\mathrm{x}' \in \bigcup_{\beta<\alpha} \mathrm{F}^{\mu}_{\beta}$ and $\mathrm{x}' \in \mathrm{F}^{\mu}_{\beta}$ for some $\beta < \alpha$.

Why two games?

Pragmatic answer Because this makes our arguments work.

Observation There seems to be no direct transformation for $\exists$'s winning strategies in the first game into winning strategies in the second game.

Even 2 games are not enough!

Four Games: Two more for greatest fixpoints.

The two greatest fixpoint games

Game $\mathcal{G}_\nu^{\mathrm{I}}$:

Position	Pl	Moves
$x \in X$	$\exists$	$\{C \subseteq X \mid x \in F(U) \text{ for all } U \in Clp(\mathbb{X}) \text{ with } Int(C) \subseteq U\}$
$C \subseteq X$	$\forall$	C

Game $\mathcal{G}_\nu^{\mathrm{II}}$:

Position	Player	Moves
$x \in X$	$\exists$	$\{U \in Clp(\mathbb{X}) \mid x \in F(U)\}$
$U \in Clp(\mathbb{X})$	$\forall$	U

evaluation game

Evaluation game

Goal

For a formula $\varphi \in \mathcal{L}_\mu$ and a model $\mathbb{M} = (\mathbb{X}, \mathrm{R}, \mathrm{V})$ define a game s.t. $\exists$ has a winning strategy at position (φ, x) iff $\mathrm{x} \in [\![\varphi]\!]^{\mathbb{M}}_{\mathrm{V}}$.

Evaluation game $\mathcal{E}(\varphi, \mathbb{M})$: Standard formulation

Given a formula $\varphi \in \mathcal{L}_\mu$ we define its evaluation game as follows:

Position	Condition	Player	Possible Moves
(p, x)	$\mathrm{p} \in \mathrm{FV}(\varphi), \mathrm{x} \notin \mathrm{V}(\mathrm{p})$	$\exists$	$\emptyset$
(p, x)	$\mathrm{p} \in \mathrm{FV}(\varphi), \mathrm{x} \in \mathrm{V}(\mathrm{p})$	$\forall$	$\emptyset$
$(\neg\mathrm{p}, \mathrm{x})$	$\mathrm{p} \in \mathrm{FV}(\varphi), \mathrm{x} \notin \mathrm{V}(\mathrm{p})$	$\forall$	$\emptyset$
$(\neg\mathrm{p}, \mathrm{x})$	$\mathrm{p} \in \mathrm{FV}(\varphi), \mathrm{x} \in \mathrm{V}(\mathrm{p})$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, \mathrm{x})$		$\forall$	$\{(\psi_1, \mathrm{x}), (\psi_2, \mathrm{x})\}$
$(\psi_1 \vee \psi_2, \mathrm{x})$		$\exists$	$\{(\psi_1, \mathrm{x}), (\psi_2, \mathrm{x})\}$
$(\Diamond\psi, \mathrm{x})$		$\exists$	$\{(\psi, \mathrm{x}') \mid \mathrm{x}' \in \mathrm{R}[\mathrm{x}]\}$
$(\Box\psi, \mathrm{x})$		$\forall$	$\{(\psi, \mathrm{x}') \mid \mathrm{x}' \in \mathrm{R}[\mathrm{x}]\}$
$(\eta\mathrm{p}.\psi, \mathrm{x})$	$\eta \in \{\mu, \nu\}$	$\exists/\forall$	(ψ, x)
(p, x)	$\mathrm{p} \in \mathrm{BV}(\varphi), \varphi@\mathrm{p} = \mu\mathrm{p}.\psi$	$\exists/\forall$	(ψ, x)

Plus some suitable winning condition on infinite plays.

Evaluation game $\mathcal{E}(\varphi, \mathbb{M})$: our modification

Given a formula $\varphi \in \mathcal{L}_\mu$ we remove the last rule and add the following rules on fixpoint unfolding:

Position	Condition	Player	Possible Moves
$(\eta p.\psi, x)$	$\eta \in \{\mu, \nu\}$	$\exists/\forall$	(ψ, x)
(p, x)	$\eta = \mu$	$\forall$	$\{(p, U) \mid U \in Clp(\mathbb{X}),\ x \in U\}$
(p, x)	$\eta = \nu$	$\exists$	$\{(p, U) \mid U \in Clp(\mathbb{X}),\ x \in U\}$
(p, U)	$\eta = \mu$	$\exists$	$\{(\psi, x') \mid x' \in U\}$
(p, U)	$\eta = \nu$	$\forall$	$\{(\psi, x') \mid x' \in U\}$

where $p \in BV(\varphi)$, $\varphi@p = \eta p.\psi$ with $\eta \in \{\mu, \nu\}$, $U \in Clp(\mathbb{X})$.

Intuition

Reachability becomes easier to prove; safety more difficult.

Bisimulations

Let $\mathbb{M}_1 = (\mathbb{X}_1, R_1, V)$ and $\mathbb{M}_2 = (\mathbb{X}_2, R_2, V)$ be extremally disconnected Kripke models.

A relation $Z \subseteq X_1 \times X_2$ is called a clopen bisimulation iff

- $Z \subseteq X_1 \times X_2$ is a (standard) Kripke bisimulation and
- for any clopen subsets $U_1 \in \mathrm{Clp}(\mathbb{X}_1)$ and $U_2 \in \mathrm{Clp}(\mathbb{X}_2)$:

$$\begin{aligned} Z[U_1] &= \{x' \in X_2 \mid \exists x \in U_1.(x, x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_2) \\ Z^{-1}[U_2] &= \{x \in X_1 \mid \exists x' \in U_2.(x, x') \in Z\} \in \mathrm{Clp}(\mathbb{X}_1). \end{aligned}$$

Bisimulation invariance

Let Z be a clopen bisimulation between extremally disconnected Kripke models $\mathbb{M}_1 = (\mathbb{X}_1, \mathrm{R}_1, \mathrm{V})$ and $\mathbb{M}_2 = (\mathbb{X}_2, \mathrm{R}_2, \mathrm{V})$.

Proposition

For any formula $\varphi \in \mathcal{L}_\mu$ and states $\mathrm{x} \in \mathrm{X}_1$ and $\mathrm{x}' \in \mathrm{X}_2$ such that $(\mathrm{x}, \mathrm{x}') \in \mathrm{Z}$, we have $\mathrm{x} \in [\![\varphi]\!]$ iff $\mathrm{x}' \in [\![\varphi]\!]$.

Proof sketch

Suppose that $(\mathrm{x}, \mathrm{x}') \in \mathrm{Z}$ and that $\mathrm{x} \in [\![\varphi]\!]$ for some formula φ.

$\Rightarrow (\varphi, \mathrm{x}) \in \mathrm{Win}_{\exists}(\mathcal{E}(\varphi, \mathbb{M}_1))$.

$\Rightarrow$ Transform $\exists$'s winning strategy in $\mathcal{G}_1 = \mathcal{E}(\varphi, \mathbb{M}_1)$ at position (φ, x) into a winning strategy for $\exists$ in $\mathcal{G}_2 = \mathcal{E}(\varphi, \mathbb{M}_2)$ at position (φ, x').

$\Rightarrow \mathrm{x}' \in [\![\varphi]\!]$

Future & Ongoing Work

- other topological spaces
- other notions of admissible subset
- automata & complexity?
- shedding light on completeness proof of fixpoint logics (modal μ-calculus)
- if (some of) this works out for Kripke frames, we could look into coalgebraic variants (game logic)

Thanks!