# Relational decision procedures with their applications to nonclassical logics

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# Outline

1. Dual tableaux – an overview
2. Relational logic and relational deduction
3. Relational decision procedures
4. Examples of applications

# Deduction systems

## Axiomatic deduction systems

Systems in Hilbert style [Frege, Russell, Heyting]:

- system: axioms (many) + rule (one)
- proof – finite sequence of formulas

## Non-Hilbertian systems

- Gentzen's calculus of sequents
- analytic tableaux – Beth 1955 and Hintikka 1955
  - Diagrams – Rasiowa and Sikorski 1960
  - Tableaux – Smullyan 1968 and Fitting 1990

Smullyan tableaux and Rasiowa-Sikorski diagrams are dual.

# Dual Tableaux – inspired by Rasiowa-Sikorski diagrams

- The rules usually have the form: $\dfrac{\Phi}{\Phi_1 \mid \ldots \mid \Phi_n}$
- ',' – disjunction '|' – conjunction
- $X$ is valid iff the meta-disjunction of formulas from $X$ is valid
- The rules are semantically invertible, that is for every set $X$ of formulas:

$$X \cup \Phi \text{ is valid iff all } X \cup \Phi_i \text{ are valid}$$

- Axioms: some valid sets of formulas
- Proof: a decomposition tree
- Provability of a formula: existence of a closed proof tree

# Dual tableau for first-order logic with identity

## Decomposition rules for connectives:

(RS$\vee$) $\dfrac{\varphi \vee \psi}{\varphi, \psi}$ (RS$\neg\vee$) $\dfrac{\neg(\varphi \vee \psi)}{\neg\varphi \mid \neg\psi}$

(RS$\neg$) $\dfrac{\neg\neg\varphi}{\varphi}$

## Decomposition rules for quantifiers:

(RS$\forall$) $\dfrac{\forall x \varphi(x)}{\varphi(z)}$ (RS$\neg\forall$) $\dfrac{\neg\forall x \varphi(x)}{\neg\varphi(z), \neg\forall x \varphi(x)}$

$z$ is a new variable $z$ is any variable

# Dual tableau for first-order logic with identity

**Specific rule for identity:**

$$(\text{RS}=) \quad \frac{\varphi(x)}{x = y, \varphi(x) \mid \varphi(y), \varphi(x)}$$

$\varphi$ is an atomic formula, $y$ is any variable

**Axiomatic sets:**

- $\varphi, \neg\varphi$
- $x = x$

# Example $\neg\forall x(\varphi \vee \psi(x)) \vee (\varphi \vee \forall x\psi(x))$

$$\underline{\neg\forall x(\varphi \vee \psi(x)) \vee (\varphi \vee \forall x\psi(x))}$$

↓ (RS∨) twice

$$\neg\forall x(\varphi \vee \psi(x)), \varphi, \underline{\forall x\psi(x)}$$

↓ (RS∀) with a new variable $z$

$$\underline{\neg\forall x(\varphi \vee \psi(x))}, \varphi, \psi(z)$$

↓ (RS¬∀) with variable $z$

$$\underline{\neg(\varphi \vee \psi(z))}, \varphi, \psi(z), \ldots$$

(RS¬∨)

| | |
|---|---|
| $\neg\varphi, \varphi, \ldots$ | $\neg\psi(z), \psi(z), \ldots$ |
| closed | closed |

The common language of most dual tableaux is

THE LOGIC RL OF BINARY RELATIONS.

## Formal features of RL

- Formulas are intended to represent statements saying that two objects are related.
- Relations are specified in the form of relational terms.
- Terms are built from relational variables and relational constants with relational operations.

# Relational logics – why?

## Formal motivation

The relational logic RL is the logical representation of

REPRESENTABLE RELATION ALGEBRAS

introduced by Tarski.

## Representable Relation Algebras RRA:

- Relation algebras that are isomorphic to proper algebras of binary relations
- Not all relation algebras are representable
- RRA is not finitely axiomatizable
- RRA is a discriminator variety with a recursively enumerable but undecidable equational theory

# Relational logics – why?

## Possible answer

- Broad applicability.
- Elements of relational structures can be interpreted as possible worlds, points (intervals) of time, states of a computer program, etc.
- We gain compositionality: the relational counterparts of the intensional connectives become compositional, that is the meaning of a compound formula is a function of meaning of its subformulas.
- It enables us to express an interaction between information about static facts and dynamic transitions between states in a single uniform formalism.

# Relational logics – why?

## Advantages of the relational logic

- A generic logic suitable for representing within a uniform formalism the three basic components of formal systems: syntax, semantics, and deduction apparatus.
- A general framework for representing, investigating, implementing, and comparing theories with incompatible languages and/or semantics.
- A great variety of logics can be represented within the relational logic, in particular modal, temporal, spatial, information, program, as well as intuitionistic, and many-valued, among others.

# Relational dual tableaux – why?

## Possible answer

Methodology of relational dual tableaux enables us to build proof systems for various theories in a systematic modular way:

- A dual tableau for the classical relational logic of binary relations is a core of most of the relational proof systems.
- For any particular logic some specific rules are designed and adjoined to the core set of rules.
- Relational dual tableau systems usually do more: they can be used for proving entailment, model checking, and satisfaction in finite models.

# Advantages of relational dual tableaux

- We need not implement each deduction system from scratch.
- We only extend the core system with a module corresponding to a specific part of a logic under consideration.

# Relational logic RL of binary relations

## Language

- object variables: $x, y, z, \ldots$
- relational variables: $P_1, P_2, \ldots$
- relational constants: $1, 1'$
- relational operations: $-, \cup, \cap, ^{-1}, ;$

## Terms and formulas

- Atomic term: a relational variable or constant
- Compound terms: $-P$, $P \cup Q$, $P \cap Q$, $P^{-1}$, $P ; Q$
- Formulas: $xTy$

# Relational logic RL

**Relational model: $\mathcal{M} = (U, m)$**

- $U$ – a non-empty set
- $m(P)$ – any binary relation on $U$
- $m(1) = U \times U$, $m(1') = Id_U$
- $m(-Q) = (U \times U) \setminus m(Q)$
- $m(Q \cup T) = m(Q) \cup m(T)$
- $m(Q \cap T) = m(Q) \cap m(T)$
- $m(Q^{-1}) = m(Q)^{-1}$
- $m(Q; T) = m(Q); m(T) =$

  $\{(x, y) \in U \times U : \exists z \in U((x, z) \in m(Q) \wedge (z, y) \in m(T))\}$.

# Relational logic RL

## Valuation

Any function $v$ that assigns object variables to elements from $U$.

## Semantics

- Satisfaction, $\mathcal{M}, v \models xTy$: $\quad (v(x), v(y)) \in m(T)$
- Truth, $\mathcal{M} \models xTy$: $\quad$ satisfaction by all valuations in $\mathcal{M}$
- Validity: truth in all models.

# Dual tableau for the relational logic RL

**Decomposition rules:**

$(\cup)\quad \dfrac{x(R \cup S)y}{xRy, xSy}$

$(-\cup)\quad \dfrac{x{-}(R \cup S)y}{x{-}Ry \mid x{-}Sy}$

$(;)\quad \dfrac{x(R;S)y}{xRz, x(R;S)y \mid zSy, x(R;S)y}$

$z$ is any variable

$(-;)\quad \dfrac{x{-}(R;S)y}{x{-}Rz, z{-}Sy}$

$z$ is a new variable

# Dual tableau for the relational logic RL

## Specific rules:

$$(1'1) \quad \frac{xRy}{xRz, xRy \mid y1'z, xRy} \qquad (1'2) \quad \frac{xRy}{x1'z, xRy \mid zRy, xRy}$$

$z$ is any object variable, $R$ is an atomic term

## Axioms:

- $xTy, x{-}Ty$
- $x1y$
- $x1'x$

# Main Results

## Soundness and Completeness

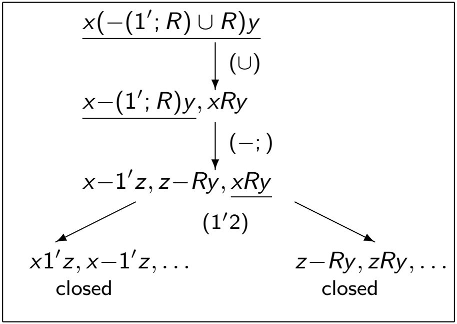
For every RL-formula $\varphi$ the following conditions are equivalent:

1. $\varphi$ is RL-valid.
2. $\varphi$ is RL-provable.

## The connection between RL and RRA

For every relational term $R$ the following conditions are equivalent, for all object variables $x$ and $y$:

- $R = 1$ is RRA-valid.
- $xRy$ is RL-valid.


$x(-(1'; R) \cup R)y$
$(\cup)$
$x-(1'; R)y, xRy$
$(-;)$
$x-1'z, z-Ry, xRy$
$(1'2)$
$x1'z, x-1'z, \ldots$
closed
$z-Ry, zRy, \ldots$
closed

# Entailment in RL

**Fact [Tarski 1941]**

$$R_1 = 1, \ldots, R_n = 1 \text{ imply } R = 1$$

iff

$$(1; -(R_1 \cap \ldots \cap R_n); 1) \cup R = 1.$$

**Entailment can be expressed in RL:**

$$xR_1y, \ \ldots, \ xR_ny \text{ imply } xRy$$

iff

$$x(1; -(R_1 \cap \ldots \cap R_n); 1) \cup R)y \text{ is RL-valid.}$$

# Model Checking and Satisfaction Problem

## Problem

Let $\mathcal{M} = (U, m)$ be a finite RL-model, $\varphi = xRy$ be an RL-formula, and $v$ be a valuation in $\mathcal{M}$.

1. Model checking: $\mathcal{M} \models \varphi$?
2. Satisfaction problem: $\mathcal{M}, v \models \varphi$?

## How to verify?

- Define the logic $\mathsf{RL}_{\mathcal{M},\varphi}$ coding $\mathcal{M}$ and $\varphi$
- Construct dual tableau for $\mathsf{RL}_{\mathcal{M},\varphi}$

For details see the book [Orłowska-Golińska-Pilarek 2011].

# Alternative versions of the relational logic RL

Most of the non-classical logics can be translated either into a fragment or an extension of the relational logic RL.

## Possible fragments of RL

- without the relational constants 1 and $1'$
- some restriction on terms built with the composition operation

## Possible extensions of RL

- with object constants and/or object operations
- more relational constants and/or relational operations
- additional $n$-ary relational symbols, for $n > 2$

Other: any combination of the above without object/relational variables (only object/relational constants).

# Relational representation of a non-classical logic L

- Development of a relational semantics for L (e.g., Kripke semantics).
- Development of a relational logic $\mathsf{RL_L}$ appropriate for a logic L.
- Development of a validity preserving translation, $\tau$, from the language of logic L into the language of logic $\mathsf{RL_L}$.
- Construction of a dual tableau for $\mathsf{RL_L}$ such that for every formula $\varphi$ of L, $\varphi$ is valid in L iff its translation $\tau(\varphi)$ is provable in $\mathsf{RL_L}$.
- Construction of a dual tableau for extensions of $\mathsf{RL_L}$ used for verification of entailment, for model checking, and for verification of satisfiability in the logic L.

# The general relational methodology

Relational dual tableaux have been constructed for a great variety of non-classical logics:

- modal, temporal, epistemic, dynamic,
- intuitionistic and relevant,
- many-valued, fuzzy, rough-set-based, among others.

**Disadvantages**

The general relational methodology does not guarantee that the constructed system will be a decision procedure.

In most cases it is NOT, while logics for which systems are constructed are decidable.

# Towards decision procedures

## Possible approaches

- Restricted relational language and/or applications of standard RL-rules that can generate infinite trees, for instance:
  - The rule (; ) cannot precede an application of the rule (−;) and a chosen variable $z$ must occur on a branch.
    (Used in systems for simple fragments of RL, see [OGP11].)
  - A relational language is restricted: only special forms of composition terms are allowed; some additional requirements on applications of standard RL-rules are assumed.
    (Used in systems for those fragments of RL that can be used to express modal and description logics. For details see papers of Cantone, Nicolosi-Asmundo, and Orłowska.)

## Possible approaches

- New rules instead of 'bad' rules.
- External techniques typical for tableaux: backtracking, backjumping, simplifications.
- Any combination of the above.

## Objective

To establish a general methodology for constructing relational decision procedures.

# Example of a promising approach

Relational decision procedures presented in the paper

J. Golińska-Pilarek, T. Huuskonen, and E. Munoz-Velasco, "Relational dual tableau decision procedures and their applications to modal and intuitionistic logics", *Annals of Pure and Applied Logics* 165(2), 2014, 409–427, doi: 10.1016/j.apal.2013.06.003

can serve as:

- decision procedures for modal and intuitionistic logics,
- a starting point for a general relational decision procedure.

# The main features of the approach

- Only restricted forms of relational terms with composition are allowed.
- New rules for the composition operation.
- New rules corresponding to specific properties of the accessibility relation.
- Additional external constraints on applications of rules.
- Exactly one finite tree for each formula.
- Each of the systems is not only a base for an algorithm verifying validity of a formula, but is ITSELF a decision procedure, with all the necessary bookkeeping built into the rules.

# A fragment of the relational logic: $\mathsf{RL}^*$

## Language of $\mathsf{RL}^*$

- object variables: $\mathbb{OV} = \{z_0, z_1, \ldots\}$
- relational variables: $\mathbb{RV} = \{P_1, P_2, \ldots\}$
- the single relational constant: $R$
- relational operations: $\{-, \cap, ;\}$.

## Relational terms of $\mathsf{RL}^*$

- Relational variables are terms.
- If $S$, $T$ are terms, then so are $-S, S \cap T, (R\,;T)$.

Relational formulas are of the form $z_n T z_0$, for $n \geq 1$.

Terms and formulas are uniquely ordered.

# A fragment of the relational logic: $\mathsf{RL}^*$

## Important feature

The relational constant $R$ and the composition operator ; are syntactically inseparable; the composition operator allows only terms with $R$ on the left.

$R$ alone is not a term.

Only the object on the left is significant in a formula; the right-hand side has the fixed dummy variable.

## Why $\mathsf{RL}^*$?

Such a restricted relational language is rich enough to express many non-classical logics, e.g., some modal and intuitionistic.

# A fragment of the relational logic: $RL^*$

## Semantics

Relational models, satisfaction, truth, and validity are defined in a standard way.

Thus, models are of the form $(U, m)$ and such that:

- Relational variables are interpreted as right ideal relations.
- $m(R)$ may satisfy some additional conditions (reflexivity, transitivity, heredity).
- $m$ satisfies the standard conditions of RL-models.

# New rules

**All the systems contain the following rules:**

- $(-)$, $(\cap)$, $(-\cap)$ – old rules in the new fashion
- $(R;)$ – the new rule for terms built with the composition operator

**Given a logic, its system may contain the rules:**

- (ref) – a new rule for reflexivity
- (tran) – a new rule for transitivity
- (her) – a new rule for heredity condition.

# Additional requirements

In the definition of a decomposition tree we additionally assume:

1. Whenever several rules are applicable to a node, the first possible schema from the following list is chosen: $(-)$, $(-\cap)$, $(\cap)$, (ref), (her), (tran), and $(R;)$.
   Within the schema, the instance with the minimal formula is applied.
2. The rule $(R;)$ can be applied to a node provided that its proper part is not a subcopy of any of its predecessor nodes.
3. On a branch the rule (ref) can be applied to a given formula at most once.

# The rule $(R;)$

$$(R;)\quad \frac{X \cup \{z_k A_m z_0 \mid m \in M\} \cup \{z_k{-}(R\,;S_i)z_0 \mid i \in I\} \cup \{z_k(R\,;T_j)z_0 \mid j \in J\}}{X \cup \{z_k A_m z_0 \mid m \in M\} \cup \{z_{k_i}{-}S_i z_0 \mid i \in I\} \cup \{z_{k_i} T_j z_0 \mid i \in I, j \in J\}}$$

1. $k \geq 1$,
2. $z_k T z_0 \notin X$,
3. $M$, $I$, $J$ are sets of indices, $I \neq \emptyset$,
4. $A_m$ is a literal and $S_i$, $T_j$ are terms,
5. $N = \{k_i \mid i \in I\}$ is the set of consecutive natural numbers that do not occur in the premise.

# Specific rules

(ref) $$\frac{X \cup \{z_k(R^s\,;T)z_0\}}{X \cup \{z_k(R^s\,;T)z_0\} \cup \{z_k(R^i\,;T)z_0 \mid i \in \{0,\ldots,s-1\}\}}$$

$T$ is a non-compositional term,

For all $t > s$, it holds that $z_k(R^t\,;T)z_0 \notin X$.

(tran) $$\frac{X \cup \{z_k(R\,;T)z_0\}}{X \cup \{z_k(R\,;T)z_0\} \cup \{z_k(R^2\,;T)z_0\}},$$

$T$ is a non-compositional term.

(her) $$\frac{X \cup \{z_k{-}(R\,;T)z_0\} \cup \{z_k{-}P_i z_0 \mid i \in I\}}{X \cup \{z_k{-}(R\,;T)z_0\} \cup \{z_k{-}P_i z_0 \mid i \in I\} \cup \{z_k(R\,;-P_i)z_0 \mid i \in I\}},$$

$z_k{-}Pz_0 \notin X$ for any relational variable $P$.

# Main theorems

**Termination and uniqueness**

Every formula has exactly one finite tree.

**Soundness and completeness**

For every formula $\varphi$:

$\varphi$ is valid if and only if $\varphi$ is provable.

# Example of applications – modal logics

$\mathsf{RL}^*$ can be applied as for the relational representation of modal logics of transitive or reflexive frames.

Let L be a modal logic. Then, a relational logic for L is $\mathsf{RL_L}^*$ determined by the following translation.

**Translation of a standard modal logic L into terms of $\mathsf{RL_L}^*$**

- $\tau(p_i) = P_i$, for any $p_i \in \mathbb{V}$, $i \geq 1$
- $\tau(\neg\varphi) = -\tau(\varphi)$
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \cap \tau(\psi)$
- $\tau(\langle R \rangle \varphi) = R\,;\tau(\varphi)$
- $\tau([R]\varphi) = -(R\,;-\tau(\varphi))$

$\mathsf{RL_L}^*$-models must satisfy all the constraints imposed on $R$ in L-models.

# Main theorems

The translation $\tau$ preserves validity:

**Translation Theorem**

For every L-formula $\varphi$:

$$\varphi \text{ is L-valid if and only if } z_1\tau(\varphi)z_0 \text{ is } \mathsf{RL_L}^*\text{-valid.}$$

Let L be a modal logic of reflexive or transitive frames.

**A dual tableau for L**

A dual tableau for $\mathsf{RL}^*$ with the rules (ref) or (tran).

Thus, we obtain:

**Deterministic decision procedures**

An $\mathsf{RL_L}^*$-dual tableau is a deterministic decision procedure for a logic L.

# Example of applications – intuitionistic logic INT

## Logic INT

- INT-language = the language of the classical propositional logic.
- INT-models are Kripke structures $(U, R, m)$ such that:
  - $R$ is a reflexive and transitive relation on $U$
  - For all $s, s' \in U$:

    (her) If $(s, s') \in R$ and $s \in m(p)$, then $s' \in m(p)$.

## Satisfaction

- $\mathcal{M}, s \models p$ iff $s \in m(p)$
- $\mathcal{M}, s \models \neg\varphi$ iff for every $s' \in U$, if $(s, s') \in R$, then $\mathcal{M}, s' \not\models \varphi$
- $\mathcal{M}, s \models (\varphi \rightarrow \psi)$ iff for every $s' \in U$, if $(s, s') \in R$ and $\mathcal{M}, s' \models \varphi$, then $\mathcal{M}, s' \models \psi$.

# Relational representation of the intuitionistic logic INT

## The relational logic $RL^*_{INT}$

- $RL^*_{INT}$-language is the $RL^*$-language,
- $RL^*_{INT}$-models are $RL^*$-models with $R$ interpreted as a reflexive and transitive relation satisfying heredity condition:

(her') If $(x, y) \in m(R)$ and $(x, z) \in m(P)$, then $(y, z) \in m(P)$.

## Translation $\iota$

$\iota(p_i) = P_i$, for every propositional variable $p_i$

$\iota(\neg\varphi) = -(R\,;\iota(\varphi))$

$\iota(\varphi \rightarrow \psi) = -(R\,;(\iota(\varphi) \cap -\iota(\psi)))$.

# Relational decision procedure for the logic INT

## Translation Theorem

For every INT-formula $\varphi$:

$$\varphi \text{ is INT-valid if and only if } z_1 \iota(\varphi) z_0 \text{ is } \mathsf{RL_{INT}}^*\text{-valid.}$$

## $\mathsf{RL}^*_{\mathsf{INT}}$-dual tableau

A dual tableau for $\mathsf{RL}^*$ with the rules (ref), (tran), (her).

## Relational decision procedure for INT

$\mathsf{RL}^*_{\mathsf{INT}}$-dual tableau is a deterministic decision procedure for INT.

# Further results

This methodology has been extended to multimodal logics with more than one accessibility relation and some description logics in the paper:

D. Cantone, J. Golińska-Pilarek, M. Nicolosi-Asmundo. A relational dual tableau decision procedure for multimodal and description logics, in: M. Polycarpou et al. (eds.), *Hybrid Artificial Intelligence Systems*, Springer, LNCS 8480, 2014, 466477.

**The most recent research**

Relational decision procedure for the qualitative modal logic of order of magnitude reasoning with distance relation – $\mathrm{OMR_D}$.

# Qualitative Reasoning

QR is an approach within Artificial Intelligence for dealing with commonsense knowledge about the physical world.

The crucial issue of QR is to represent and reason about continuous properties of objects in a symbolic but human-like manner; with no reliance on numerical information.

Given a context, qualitative representation makes only as many distinctions as necessary to identify objects, events, situations, etc.

It is an adequate tool for dealing with situations in which information is not sufficiently precise (e.g., numerical values are not available).

# Motivations for QR

Human knowledge is almost always incomplete.

- People often draw useful conclusions about the real world without mathematical equations or theories.
- They figure out what is happening and how they can affect it, even if they have less precise data than would be required to use traditional, purely quantitative and numerical methods.
- Scientists use qualitative reasoning when they initially try to understand a problem, when they set up formal representation for a particular task, and when they interpret quantitative calculation or simulation.

Order-of-magnitude Reasoning (OMR) is an approach within QR.

The order-of-magnitude approach enables us to reason in terms of relative magnitudes of variables obtained by comparisons of the sizes of quantities.

OMR methods of reasoning are situated midway between numerical methods and purely qualitative formalisms.

# Order-of-magnitude reasoning

OMR-approaches:

- Absolute Order of Magnitude (AOM) – represented by a partition of the real line $\mathbb{R}$, where each element of $\mathbb{R}$ belongs to a qualitative class.
- Relative Order of Magnitude (ROM) – represented by a family of binary order-of-magnitude relations which establish different comparison relations in $\mathbb{R}$ (e.g., comparability, negligibility or closeness).
- Both approaches, absolute and relative, can be combined.

# Logics for order-of-magnitude reasoning

Multimodal hybrid logics that enable us to deal with different qualitative relations based on qualitative classes obtained by dividing the real line in intervals.

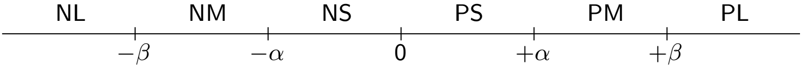
OMR-logics that have been studied include qualitative relations:

- comparability
- negligibility
- bidirectional negligibility
- non-closeness
- distance.

# $\mathrm{OMR_D}$-logic with distance

The logic $\mathrm{OMR_D}$ is based on the model AOM(5) in which the real line is divided into

*seven* equivalences classes with *five* landmarks.




where $<$ is a strict linear order on real numbers and $\alpha < \beta$.

**Distance relation D on $(U, <)$**

For all $x, y, z, x', y' \in U$,

If $x\mathrm{D}y$, then $x < y$.

$c_i\mathrm{D}c_{i+1}$, for $i \in \{1, 2, 3, 4\}$.

If $x\mathrm{D}y$ and $x\mathrm{D}z$, then $y = z$.

If $x\mathrm{D}y$, $x'\mathrm{D}y'$, and $x < x'$ then $y < y'$.

# Logic $\mathrm{OMR_D}$ – language

$\mathrm{OMR_D}$ – the multimodal logic with constants over two basic accessibility relations R and D together with their converses.

## Vocabulary

- Propositional variables: $p_1, p_2, p_3, \ldots,$
- Propositional constants: $c_1, \ldots, c_5,$
- Classical propositional operations: $\neg, \vee, \wedge, \rightarrow,$
- Modal operations: $[R], [\overline{R}], [D], [\overline{D}]$.

Formulas are defined as usual in modal logics.

# Logic $\mathrm{OMR_D}$ – axiomatization

## Axioms for landmarks

For $i \in \{1, \ldots, 5\}$ and $j \in \{1, \ldots, 4\}$

$\langle \overline{R} \rangle c_i \vee c_i \vee \langle R \rangle c_i$

$c_i \to ([\overline{R}]\neg c_i \wedge [R]\neg c_i)$

$c_j \to \langle D \rangle c_{j+1}$

## Axioms for converses and ordering

For $T \in \{R, \overline{R}, D, \overline{D}\}$ and $S \in \{R, \overline{R}\}$,

$[T](\varphi \to \psi) \to ([T]\varphi \to [T]\psi)$,

$\varphi \to [T]\langle T' \rangle \varphi$, where $T'$ is the converse of $T$,

$[R]\varphi \to [R][R]\varphi$,

$[S]([S]\varphi \to \psi) \vee [S]([S]\psi \to \varphi)$

# Logic $OMR_D$ – axiomatization

## Axioms for distance relation

$[R]\varphi \to [D]\varphi$

$\langle D\rangle\varphi \to [D]\varphi$

$(\varphi \wedge \langle D\rangle\psi \wedge \langle R\rangle(\chi \wedge \langle D\rangle\theta)) \leftrightarrow \langle R\rangle(\theta \wedge \langle \overline{D}\rangle\chi \wedge \langle \overline{R}\rangle(\psi \wedge \langle \overline{D}\rangle))$

## Rules of inference

If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.

If $\vdash \varphi$, then $\vdash [R]\varphi$.

If $\vdash \varphi$, then $\vdash [\overline{R}]\varphi$.

Provability of a formula is defined in a standard way.

# Logic $\mathsf{OMR_D}$ – models

Structures of the form $\mathcal{M} = (U, \mathsf{R}, \overline{\mathsf{R}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{c}_1, \ldots, \mathsf{c}_5, m)$, where:

- $U$ – a nonempty set,
- $\mathsf{R}$ is a strict linear order on $U$ and $\overline{\mathsf{R}}$ is the converse of $\mathsf{R}$,
- $\mathsf{D} \subseteq \mathsf{R}$ and $\overline{\mathsf{D}}$ is the converse of $\mathsf{D}$,
- $\mathsf{D}$ is partially functional and satisfies:

  If $s\mathsf{D}t$, $s'\mathsf{D}t'$, $s\mathsf{R}s'$, then $t\mathsf{R}t'$, for all $s, s', t, t' \in U$,
- $m(p) \subseteq U$, for every propositional variable $p$
- $m(c_i) = \mathsf{c}_i \in U$ and $\mathsf{c}_i \neq \mathsf{c}_j$, for all $i, j \in \{1, \ldots, 5\}$, $i \neq j$,
- $(\mathsf{c}_i, \mathsf{c}_{i+1}) \in \mathsf{D}$, for all $i \in \{1, \ldots, 4\}$.

# Logic $OMR_D$ – semantics

## Semantics

Satisfaction: defined as usual in modal logics.

Truth in a model: satisfaction by all states.

$OMR_D$-validity: truth in all models.

## Soundness and Completeness

For every formula $\varphi$:

$$\varphi \text{ is } OMR_D\text{-provable iff } \varphi \text{ is } OMR_D\text{-valid.}$$

For details see [Burrieza et al. 2007] and [Zawidzki 2017].

# Relational representation of $\mathrm{OMR_D}$ – $\mathrm{RL_D}$

## Language

- $z_0, z_1, \ldots$ – object variables,
- $P_1, P_2, \ldots$ – relational variables,
- $C_1, \ldots, C_5$ – relational constants representing propositional constants from $\mathrm{OMR_D}$,
- $R, \overline{R}, D, \overline{D}$ – relational constants representing accessibility relations of $\mathrm{OMR_D}$,,
- $-, \cap, ;$ – relational operations.

## Relational terms

- Relational variables and $C_1, \ldots, C_5$ are terms.
- If $S$, $T$ are terms and $r \in \{R, \overline{R}, D, \overline{D}\}$, then so are $-S, S \cap T, (r\,;T)$.

Formulas: $z_i T z_0$, for $i \geq 1$ and a relational term $T$.

# Models of $\mathsf{RL_D}$

Structures of the form $\mathcal{M} = (U, \mathsf{R}, \overline{\mathsf{R}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{C}_1, \ldots, \mathsf{C}_5, m)$, where:

(i) $U$ – a nonempty set,

(ii) $m(P) = X \times U$, where $X \subseteq U$, for every relational variable $P$,

(iii) $m(C_i) = \mathsf{C}_i \subseteq X \times U$, where $X \subseteq U$, for every $i \in \{1, \ldots, 5\}$,

(iv) $\mathsf{C}_i \cap \mathsf{C}_j = \emptyset$, for all $i, j \in \{1, \ldots, 5\}$ such that $i \neq j$,

(v) $\mathsf{R}, \mathsf{D} \subseteq U^2$, $\overline{\mathsf{R}}$ and $\overline{\mathsf{D}}$ are converses of $\mathsf{R}$ and $\mathsf{D}$, respectively, and $m(R) = \mathsf{R}$, $m(D) = \mathsf{D}$, $m(\overline{R}) = \overline{\mathsf{R}}$, $m(\overline{D}) = \overline{\mathsf{D}}$,

(vi) For all $x, y \in U$ and $i \in \{1, \ldots, 4\}$, if $(x, y) \in \mathsf{C}_i$, then there is $z \in U$ such that $(x, z) \in \mathsf{D}$ and $(z, y) \in \mathsf{C}_{i+1}$,

(vii) For all $x, y \in U$ and $i \in \{1, \ldots, 5\}$, if $(x, y) \in \mathsf{C}_i$, then for all $z \in U$, if $(x, z) \in \mathsf{R}$ or $(z, x) \in \mathsf{R}$, then $(z, y) \notin \mathsf{C}_i$,

(viii) For all $x, y \in U$ and $i \in \{1, \ldots, 5\}$, if $(x, y) \notin \mathsf{C}_i$, then either there is $z \in U$ such that both $(x, z) \in \mathsf{R}$ and $(z, y) \in \mathsf{C}_i$ or there is $z \in U$ such that both $(z, x) \in \mathsf{R}$ and $(z, y) \in \mathsf{C}_i$, Further conditions:

(ix) $\mathsf{D} \subseteq \mathsf{R}$

(x) $\mathsf{R}$ is transitive and weakly connected,

(xi) $\overline{\mathsf{R}}$ is weakly connected,

(xii) $\mathsf{D}$ and $\overline{\mathsf{D}}$ are partially functional,

(xiii) For all $x, x', y, y' \in U$, if $(x, x') \in \mathsf{D}$ and $(y, y') \in \mathsf{D}$ and $(x, y) \in \mathsf{R}$, then $(x', y') \in \mathsf{R}$,

(xiv) For all $x, x', y, y' \in U$, if $(x, x') \in \overline{\mathsf{D}}$ and $(y, y') \in \overline{\mathsf{D}}$ and $(x, y) \in \overline{\mathsf{R}}$, then $(x', y') \in \overline{\mathsf{R}}$,

(xv) $m$ extends to all the compound terms as usual.

# Relational representation of $\mathsf{OMR_D}$

## Translation of $\mathsf{OMR_D}$-formulas into $\mathsf{RL_D}$-terms

- $\tau(p_i) = P_i$, for any $p_i \in \mathbb{V}$, $i \geq 1$,
- $\tau(c_i) = C_i$, for every $i \in \{1, \ldots, 5\}$,
- $\tau(\neg\varphi) = -\tau(\varphi)$,
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \cap \tau(\psi)$,

For every $r \in \{R, \overline{R}, D, \overline{D}\}$,

- $\tau(\langle r \rangle \varphi) = r ; \tau(\varphi)$,
- $\tau([r]\varphi) = -(r ; -\tau(\varphi))$.

Given the weak semantics for $\mathsf{OMR_D}$ defined by Zawidzki in [Zaw17], it can be proved the following:

## Translation theorem

For every $\mathsf{OMR_D}$-formula $\varphi$:

$$\varphi \text{ is } \mathsf{OMR_D}\text{-valid iff } z_1 \tau(\varphi) z_0 \text{ is } \mathsf{RL_D}\text{-valid}$$

# $\mathsf{RL_D}$-dual tableau

A dual tableau for $\mathsf{RL_D}$ consists of the following rules:

- the rules $(-)$, $(\cap)$, $(-\cap)$ of $\mathsf{RL}^*$-dual tableau,
- the rule for composition $(r;)$, for $r \in \{R, \overline{R}, D, \overline{D}\}$, of $\mathsf{RL}^*$-dual tableau adjusted to $\mathsf{RL_D}$-language,
- rules for converse relations $(R\overline{R})$, $(\overline{R}R)$, $(D\overline{D})$, $(\overline{D}D)$,
- rules for constants $C_i$: (empty), (ord), $(\mathrm{irref}_1)$, $(\mathrm{irref}_2)$, (con),
- the rules for relations R, D, and their converses: $(\mathrm{DR}_1)$, $(\mathrm{DR}_2)$, (tran), (wcon), $(\mathrm{pfun_D})$, $(\mathrm{pfun_{\overline{D}}})$, $(\mathrm{dist_D})$, $(\mathrm{dist_{\overline{D}}})$.

# Examples of new rules

## Rules for relational constants $C_i$, $i, j \in \{1, \ldots, 5\}$, $i \neq j$

(empty) $$\frac{X \cup \{z_n - C_i z_0\}}{X \cup \{z_n - C_i z_0, z_n C_j z_0\}}$$

(ord) $$\frac{X \cup \{z_n (D; C_{i+1}) z_0\}}{X \cup \{z_n (D; C_{i+1}) z_0, z_n C_i z_0\}}$$

($\text{irref}_1$) $$\frac{X \cup \{z_n - (R; C_i) z_0\}}{X \cup \{z_n - (R; C_i) z_0, z_n C_i z_0\}}$$

($\text{irref}_2$) $$\frac{X \cup \{z_n - (\overline{R}; C_i) z_0\}}{X \cup \{z_n - (\overline{R}; C_i) z_0, z_n C_i z_0\}}$$

(con) $$\frac{X \cup \{z_n (\overline{R}; C_i) z_0, z_n (R; C_i) z_0\}}{X \cup \{z_n (\overline{R}; C_i) z_0, z_n (R; C_i) z_0, z_n - C_i z_0\}}$$

## Rules for the condition D ⊆ R

($\text{DR}_1$) $$\frac{X \cup \{z_n (R; T) z_0\}}{X \cup \{z_n (R; T) z_0, z_n (D; T) z_0\}}$$

($\text{DR}_2$) $$\frac{X \cup \{z_n - (D; T) z_0\}}{X \cup \{z_n - (D; T) z_0, z_n - (R; T) z_0\}}$$

# Examples of new rules

## Rules for weak connectedness

For $r \in \{R, \overline{R}\}$

(wcon) $\dfrac{X \cup G}{X \cup G \cup \{z_n(r;T)z_0\} \mid X \cup G \cup \{z_n(r;T')z_0\}}$

where

$G = \{z_n(r;(T \cap (r;T')))z_0, z_n(r;(T \cap T'))z_0, z_n(r;((r;T) \cap T'))z_0\}$

## Rules for partial functionality

$(\mathrm{pfun}_{\mathrm{D}})$ $\dfrac{X \cup \{z_n -(D;T)z_0\}}{X \cup \{z_n -(D;T)z_0, z_n(D;-T)z_0\}}$

$(\mathrm{pfun}_{\overline{\mathrm{D}}})$ $\dfrac{X \cup \{z_n -(\overline{D};T)z_0\}}{X \cup \{z_n -(\overline{D};T)z_0, z_n(\overline{D};-T)z_0\}}$

# Examples of new rules

## Rules for the distance condition

(dist$_{\mathrm{D}}$) $$\frac{X \cup H_D}{X \cup H_D \cup \{z_n(D;T)z_0\} \mid X \cup H_D \cup \{z_n(R;(D;T'))z_0\}}$$

where $H_D = \{z_n(R;(T \cap (R;T')))z_0\}$

(dist$_{\overline{\mathrm{D}}}$) $$\frac{X \cup H_{\overline{D}}}{X \cup H_{\overline{D}} \cup \{z_n(\overline{D};T)z_0\} \mid X \cup H_{\overline{D}} \cup \{z_n(\overline{R};(\overline{D};T'))z_0\}}$$

where $H_{\overline{D}} = \{z_n(\overline{R};(T \cap (\overline{R};T')))z_0\}$

## Order on applications of the rules

($-$), ($-\cap$), ($\cap$), (empty), (ord), (irref$_1$), (irref$_2$), (con), (DR$_1$), (DR$_2$), (tran), (wcon), (pfun$_{\mathrm{D}}$), (pfun$_{\overline{\mathrm{D}}}$), (dist$_{\mathrm{D}}$), (dist$_{\overline{\mathrm{D}}}$), ($R\overline{R}$), ($\overline{R}R$), ($D\overline{D}$), ($\overline{D}D$), (tran)

Finally:

the rules $(r;)$ – all compositions are decomposed at the same time

# Some open problems

- Can be this approach extended to modal logics with sufficience and dual sufficiency operators?
- Can be this approach extended to other non-classical logics?
- Is there any other general and modular way to construct a relational decision procedure?

# Thank you!

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